

SLOCC classification of n qubits invoking the proportional relationships for spectrums and for standard Jordan normal forms

Dafa Li^{1,2}

¹*Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China*

²*Center for Quantum Information Science and Technology,*

Tsinghua National Laboratory for Information Science and Technology (TNList), Beijing, 100084, China

We investigate the proportional relationships for spectrums and for SJNFs (Standard Jordan Normal Forms) of the matrices constructed from coefficient matrices of two SLOCC (stochastic local operations and classical communication) equivalent states of n qubits. Invoking the proportional relationships for spectrums and for SJNFs, pure states of n (≥ 4) qubits are partitioned into 12 groups and 34 families under SLOCC, respectively. Specially, it is true for four qubits.

I. INTRODUCTION

Quantum entanglement is an essential resource in quantum teleportation, quantum cryptography, and quantum information and computation [1]. A key task of the entanglement theory is to classify different types of entanglement. SLOCC classification is very significant because the states in the same SLOCC class are able to perform the same QIT-tasks [2][3]. It is well known that two-qubit states were partitioned into two SLOCC classes, three-qubit states were partitioned into six SLOCC classes, and there are infinitely many SLOCC classes for n (≥ 4) qubits [2]. It is highly desirable to partition these infinite classes into a finite number of families according to a SLOCC invariant criterion. In the pioneering work of Verstraete *et al.* [3], by using their general singular value decomposition Verstraete *et al.* partitioned pure four-qubit states into nine SLOCC inequivalent families: G_{abcd} , L_{abc_2} , $L_{a_2b_2}$, L_{ab_3} , L_{a_4} , $L_{a_20_3\oplus 1}$, $L_{0_5\oplus 3}$, $L_{0_7\oplus 1}$, and $L_{0_3\oplus 10_3\oplus 1}$ [3]. Since then, the extensive efforts have contributed to studying entanglement classification of four qubits [3–13].

Recently, considerable efforts have been devoted to find SLOCC invariant polynomials in the coefficients of states for classifications and measures of entanglement of n qubits [11, 14–22]. It is well known that the concurrence and the 3-tangle are invariant polynomials of degrees 2 and 4 for two and three qubits, respectively [23]. Explicit and simple expresses of invariant polynomials of degrees 2 for even n qubits [20], 4 for odd n (≥ 4) qubits [20], 4 for even n (≥ 4) qubits [21], were presented.

Very recently, SLOCC invariant ranks of the coefficient matrices were proposed for SLOCC classification [24–28].

In this paper, for two SLOCC equivalent states of n qubits, we show that the matrices constructed from coefficient matrices of the two states have pro-

portional spectrums and proportional SJNFs. Invoking the proportional relationships for spectrums pure states of n (≥ 4) qubits are partitioned into 12 groups under SLOCC, and invoking the proportional relationships for SJNFs pure states of n (≥ 4) qubits are partitioned into 34 families under SLOCC. Specially, for four qubits, we obtain new SLOCC classifications.

II. SLOCC CLASSIFICATION OF n QUBITS

A. The proportional relationships for spectrums and for SJNFs

Let $|\psi\rangle = \sum_{i=0}^{2^n-1} a_i|i\rangle$ be an n -qubit pure state. It is well known that two n -qubit pure states $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC equivalent if and only if there are invertible local operators $\mathcal{A}_i \in GL(2, C)$, $i = 1, \dots, n$, such that [2]

$$|\psi'\rangle = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \dots \otimes \mathcal{A}_n |\psi\rangle. \quad (1)$$

To any state $|\psi\rangle$ of n qubits, we associate a 2^ℓ by $2^{n-\ell}$ matrix $C_{q_1 \dots q_\ell}^{(n)}(|\psi\rangle)$ whose entries are the coefficients $a_0, a_1, \dots, a_{2^n-1}$ of the state $|\psi\rangle$, where q_1, \dots, q_ℓ are chosen as the row bits [24, 25]. In [28], in terms of the coefficient matrix $C_{q_1, \dots, q_i}^{(n)}$ we constructed a 2^i by 2^i matrix $\Omega_{q_1, \dots, q_i}^{(n)}$

$$\begin{aligned} & \Omega_{q_1, \dots, q_i}^{(n)}(|\psi\rangle) \\ &= C_{q_1, \dots, q_i}^{(n)}(|\psi\rangle) v^{\otimes (n-i)} (C_{q_1, \dots, q_i}^{(n)}(|\psi\rangle))^t, \end{aligned} \quad (2)$$

where $v = \sqrt{-1}\sigma_y$ and σ_y is the Pauli operator, and C^t is the transpose of C .

From [28], when q_1 and q_2 are chosen as the row bits, we can show that if n -qubit states $|\psi'\rangle$ and $|\psi\rangle$

are SLOCC equivalent, then

$$\begin{aligned} & \Omega_{q_1 q_2}^{(n)}(|\psi'\rangle) \\ &= (\prod_{\ell=3}^n \det \mathcal{A}_{q_\ell}) (\mathcal{A}_{q_1} \otimes \mathcal{A}_{q_2}) \Omega_{q_1 q_2}^{(n)}(|\psi\rangle) (\mathcal{A}_{q_1} \otimes \mathcal{A}_{q_2})^t. \end{aligned} \quad (3)$$

Let the unitary matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix}. \quad (4)$$

Let $G_1 = T(\mathcal{A}_{q_1} \otimes \mathcal{A}_{q_2})T^+$, where T^+ is the Hermitian transpose of T . It is easy to check that $G_1 G_1^t = (\det \mathcal{A}_{q_1} \det \mathcal{A}_{q_2}) I$. Let $S_{q_1 q_2}^{(n)}(\psi') = T \Omega_{q_1 q_2}^{(n)}(|\psi'\rangle) T^t$. Then, from Eq. (3) we obtain

$$\begin{aligned} & S_{q_1 q_2}^{(n)}(\psi') \\ &= (\prod_{\ell=3}^n \det \mathcal{A}_{q_\ell}) \times \\ & T(\mathcal{A}_{q_1} \otimes \mathcal{A}_{q_2}) T^+ T \Omega_{q_1 q_2}^{(n)}(|\psi\rangle) T^t T^* (\mathcal{A}_{q_1} \otimes \mathcal{A}_{q_2})^t T^t \\ &= k G_1 S_{q_1 q_2}^{(n)}(\psi) G_1^{-1} \end{aligned} \quad (5)$$

where T^* is a conjugate matrix, $T^t T^* = I$, and $k = \prod_{\ell=1}^n \det \mathcal{A}_\ell$. Note that $S_{q_1 q_2}^{(n)}(\psi')$ and $S_{q_1 q_2}^{(n)}(\psi)$ are 4 by 4 matrices.

In this paper, we write the direct sum of standard Jordan blocks $J_{n_1}(\lambda_1), \dots$, and $J_{n_j}(\lambda_j)$ as $J_{n_1}(\lambda_1) \cdots J_{n_j}(\lambda_j)$. The Jordan block $J_1(a)$ is simply written as a . We define that the two SJNFs $J_{n_1}(\lambda_1) \cdots J_{n_j}(\lambda_j)$ and $J_{n_1}(k\lambda_1) \cdots J_{n_j}(k\lambda_j)$, where $k \neq 0$, are proportional.

Eq. (5) leads to the following theorem 1.

Theorem 1. If the states $|\psi'\rangle$ and $|\psi\rangle$ of n qubits satisfy Eq. (1), i.e. the state $|\psi'\rangle$ is SLOCC equivalent to $|\psi\rangle$, then

(1) if $S_{q_1 q_2}^{(n)}(\psi)$ has the spectrum $\lambda_1, \dots, \lambda_4$, then $S_{q_1 q_2}^{(n)}(\psi')$ has the spectrum $k\lambda_1, \dots, k\lambda_4$, where $k = \prod_{\ell=1}^n \det \mathcal{A}_\ell$.

(2) if $S_{q_1 q_2}^{(n)}(\psi)$ has the SJNF $J_{n_1}(\lambda_1) \cdots J_{n_j}(\lambda_j)$, then $S_{q_1 q_2}^{(n)}(\psi')$ has the SJNF $J_{n_1}(k\lambda_1) \cdots J_{n_j}(k\lambda_j)$, where $k = \prod_{\ell=1}^n \det \mathcal{A}_\ell$.

We give our argument as follows. Let $\Gamma = G_1 S_{q_1 q_2}^{(n)}(\psi) G_1^{-1}$. Then, $S_{q_1 q_2}^{(n)}(\psi') = k\Gamma$. Clearly, Γ is similar to $S_{q_1 q_2}^{(n)}(\psi)$. Therefore, Γ and $S_{q_1 q_2}^{(n)}(\psi)$ have the same spectrum and SJNF.

(1). It is clear that if Γ has the spectrum $\lambda_1, \dots, \lambda_4$, then $S_{q_1 q_2}^{(n)}(\psi')$ has the spectrum $k\lambda_1, \dots, k\lambda_4$.

(2). There is an invertible matrix H such that $\Gamma = H J H^{-1}$, where the SJNF $J = J_{n_1}(\lambda_1) \cdots J_{n_j}(\lambda_j)$.

Then, $k\Gamma = H k J H^{-1}$. It is not hard to see that the SJNF of kJ is $J_{n_1}(k\lambda_1) \cdots J_{n_j}(k\lambda_j)$.

Example 1. We have the following SLOCC equivalent states of four qubits: $L_{a_4}(a \neq 0)$ and $L_{a_4}(a = 1)$ [9]; $L_{a_2 0_3 \oplus 1}(a \neq 0)$ and $L_{a_2 0_3 \oplus 1}(a = 1)$ [9]; and $L_{ab_3}^*(a = 0)$ and $L_{ab_3}(a = 0)$ [25]. We list the SJNFs of $S_{1,2}^{(4)}$ of the states in Table I.

TABLE I. SJNFs of SLOCC equivalent states

state	$L_{a_4}(a \neq 0)$	$L_{a_4}(a = 1)$	$k = a^2$
SJNF	$J_4(a^2)$	$J_4(1)$	
State	$L_{a_2 0_3 \oplus 1}(a \neq 0)$	$L_{a_2 0_3 \oplus 1}(a = 1)$	$k = a^2$
SJNF	$J_2(a^2) J_2(0)$	$J_2(1) J_2(0)$	
State	$L_{ab_3}^*(a = 0)$	$L_{ab_3}(a = 0)$	$k = 1$
SJNF	$0b^2 J_2(0)$	$0b^2 J_2(0)$	

Example 2. For four qubits, let $\zeta_4 = a(|0\rangle + |15\rangle) + b(|5\rangle + |10\rangle) + |6\rangle$, and $\zeta_5 = b(|0\rangle + |15\rangle) + a(|5\rangle + |10\rangle) + |6\rangle$, where $a \neq b$. The SJNF of $S_{1,2}^{(4)}(\zeta_4)$ is $J_2(b)aa$ while the SJNF of $S_{1,2}^{(4)}(\zeta_5)$ is $J_2(a)bb$. So, by (2) of Theorem 1 the two states ζ_4 and ζ_5 are SLOCC inequivalent.

We can rewrite $S_{q_1 q_2}^{(n)}(\psi)$ as

$$S_{q_1 q_2}^{(n)}(\psi) = [T C_{q_1 q_2}^{(n)}(|\psi\rangle)] v^{\otimes(n-2)} [T C_{q_1 q_2}^{(n)}(|\psi\rangle)]^t. \quad (6)$$

B. Partition pure states of $n (\geq 4)$ qubits into 12 groups and 34 families

Theorem 1 permits a reduction of SLOCC classification of $n (\geq 4)$ qubits to a classification of 4 by 4 complex matrices. For 4 by 4 matrices, a calculation yields 12 types of CPs (characteristic polynomials), 12 types of spectrums, and 34 types of SJNFs in Table II. It is easy to see that CPs and spectrums have the same effect for SLOCC classification.

Note that in Table II, $\sigma_i \neq 0$, $\sigma_i \neq \sigma_j$ when $i \neq j$. Next, we give 12 types of CPs of 4 by 4 matrices as follows.

CP₁ : $(\sigma - \sigma_1)^4$; CP₂ : $(\sigma - \sigma_1)(\sigma - \sigma_2)^3$; CP₃ : $(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma - \sigma_3)^2$; CP₄ : $(\sigma - \sigma_1)^2(\sigma - \sigma_2)^2$; CP₅ : $\prod_{i=1}^4 (\sigma - \sigma_i)$; CP₆ : $\sigma(\sigma - \sigma_1)^3$; CP₇ : $\sigma(\sigma - \sigma_1)(\sigma - \sigma_2)^2$; CP₈ : $\sigma \prod_{i=1}^3 (\sigma - \sigma_i)$; CP₉ : $\sigma^2(\sigma - \sigma_1)^2$; CP₁₀ : $\sigma^2(\sigma - \sigma_1)(\sigma - \sigma_2)$; CP₁₁ : $\sigma^3(\sigma - \sigma_1)$; CP₁₂ : σ^4 .

For each state of $n (\geq 4)$ qubits, the spectrum of $S_{q_1 q_2}^{(n)}$ must belong to one of the 12 types of the spectrums in Table II. Let the states of $n (\geq 4)$ qubits, for which spectrums of $S_{q_1 q_2}^{(n)}$ possess the same type in

Table II, belong to the same group. Thus, the states of n (≥ 4) qubits are partitioned into 12 groups. In light of Theorem 1, the states belonging to different groups must be SLOCC inequivalent.

For each state of n (≥ 4) qubits, the SJNF of $S_{q_1 q_2}^{(n)}$ up to the order of the standard Jordan blocks must belong to one of the 34 types of the SJNFs in Table II. Let the states of n (≥ 4) qubits with the same type of SJNFs of $S_{q_1 q_2}^{(n)}$ in Table II up to the order of the Jordan blocks belong to the same family. Thus, we partition the states of n (≥ 4) qubits into 34 families. In light of Theorem 1, the states belonging to different families must be SLOCC inequivalent.

TABLE II. 12 types of CPs, 12 types of spectrums, 34 types of the SJNFs for 4 by 4 matrices, and the corresponding states for four qubits.

CP _i ;spectrum	SJNF	state	SJNF	state
1; $\sigma_1\sigma_1\sigma_1\sigma_1$	$J_4(\sigma_1)$	τ_1	$J_2(\sigma_1)J_2(\sigma_1)$	η_1
	$J_3(\sigma_1)\sigma_1$	θ_1	$\sigma_1\sigma_1J_2(\sigma_1)$	ζ_1
	$\sigma_1\sigma_1\sigma_1\sigma_1$	G_1		
2; $\sigma_1\sigma_2\sigma_2\sigma_2$	$\sigma_1J_3(\sigma_2)$	θ_2	$\sigma_1\sigma_2J_2(\sigma_2)$	ζ_2
	$\sigma_1\sigma_2\sigma_2\sigma_2$	G_2		
3; $\sigma_1\sigma_2\sigma_3\sigma_3$	$\sigma_1\sigma_2J_2(\sigma_3)$	ζ_3	$\sigma_1\sigma_2\sigma_3\sigma_3$	G_3
4; $\sigma_1\sigma_1\sigma_2\sigma_2$	$\sigma_1\sigma_1\sigma_2\sigma_2$	G_4	$\sigma_1\sigma_1J_2(\sigma_2)$	ζ_4
	$J_2(\sigma_1)J_2(\sigma_2)$	η_2		
5; $\sigma_1\sigma_2\sigma_3\sigma_4$	$\sigma_1\sigma_2\sigma_3\sigma_4$	G_5		
6; $0\sigma_1\sigma_1\sigma_1$	$0J_3(\sigma_1)$	θ_3	$0J_2(\sigma_1)\sigma_1$	ζ_6
	$0\sigma_1\sigma_1\sigma_1$	G_6		
7; $0\sigma_1\sigma_2\sigma_2$	$0\sigma_1J_2(\sigma_2)$	ζ_7	$0\sigma_1\sigma_2\sigma_2$	G_7
8; $0\sigma_1\sigma_2\sigma_3$	$0\sigma_1\sigma_2\sigma_3$	G_8		
9; $00\sigma_1\sigma_1$	$J_2(0)J_2(\sigma_1)$	κ_1	$J_2(0)\sigma_1\sigma_1$	μ_2
	$00J_2(\sigma_1)$	ζ_9	$00\sigma_1\sigma_1$	ζ_8
10; $00\sigma_1\sigma_2$	$J_2(0)\sigma_1\sigma_2$	μ_1	$00\sigma_1\sigma_2$	ζ_{10}
11; $000\sigma_1$	$J_3(0)\sigma_1$	ξ_1	$J_2(0)0\sigma_1$	θ_4
	$000\sigma_1$	ζ_{11}		
12; 0000	$J_4(0)$	$L_{07\oplus 1}$	$J_3(0)0$	ξ_2
	$J_2(0)J_2(0)$	τ_2	$J_2(0)00$	θ_5
	0000	ζ_{12}		

Example 3. For the maximally entangled states $|\Psi_2\rangle, |\Psi_4\rangle - |\Psi_6\rangle$ of five qubits and $|\Xi_2\rangle, |\Xi_4\rangle - |\Xi_7\rangle$ of six qubits [29], SJNFs of $S_{1,2}^{(5)}$ partition $|\Psi_2\rangle, |\Psi_4\rangle - |\Psi_6\rangle$ into three families, and SJNFs of $S_{1,2}^{(6)}$ partition $|\Xi_2\rangle, |\Xi_4\rangle - |\Xi_7\rangle$ into four families. See Table III.

TABLE III. SJNFs of $S_{1,2}^{(5)}$ and $S_{1,2}^{(6)}$.

states	$ \Psi_2\rangle$	$ \Psi_4\rangle$	$ \Psi_5\rangle$	$ \Psi_6\rangle$	
SJNFs	$\pm\frac{1}{2}00$	0000	0000	$0J_3(0)$	
states	$ \Xi_2\rangle$	$ \Xi_4\rangle$	$ \Xi_5\rangle$	$ \Xi_6\rangle$	$ \Xi_7\rangle$
SJNFs	$(\frac{1}{2})(\frac{1}{2})00$	0000	0000	$00J_2(0)$	$0J_3(0)$

III. SLOCC CLASSIFICATION OF TWO, THREE, AND FOUR QUBITS

A. SLOCC classification of four qubits

For four qubits, invoking the fact that $T^+T^* = v^{\otimes 2}$ Eq. (6) reduces to

$$S_{q_1 q_2}^{(4)}(\psi) = (TC_{q_1 q_2}^{(4)} T^+)(TC_{q_1 q_2}^{(4)} T^+)^t. \quad (7)$$

From the above discussion, in light of Theorem 1 pure states of four qubits are partitioned into 12 groups and 34 families in Table II. Furthermore, for each type of spectrums, CPs, and SJNFs in Table II, we give a state in Table II and the appendix for which $S_{1,2}^{(4)}$ has the corresponding type. For example, $S_{1,2}^{(4)}$ of the state θ_1 has the spectrum $\sigma_1, \sigma_1, \sigma_1, \sigma_1$, the CP $(\sigma - \sigma_1)^4$, and the SJNF $J_3(\sigma_1)\sigma_1$. It is plain to see that 12 groups and 34 families in Table II are both complete for four qubits.

Here, we make a comparison to Verstraete et al.'s nine families. They showed that for a complex n by n matrix, there are complex orthogonal matrices O_1 and O_2 such that $R = O_1 R' O_2$, where R' is a direct sum of blocks defined in [3]. Note that the blocks are not standard Jordan blocks. The decomposition was called a generalization of the singular value decomposition and used to partition pure states of four qubits into nine families [3].

Recently, Chterental and Djoković pointed out an error in Verstraete et al.'s nine families by indicating that the family L_{ab3} is SLOCC equivalent to the subfamily L_{abc_2} ($a = c$) of the family L_{abc_2} [7][28]. Thus, the classification for the nine families is incomplete. The need to redo this classification of four qubits was proposed [7].

B. SLOCC classification of three qubits

For three qubits, Eq. (6) reduces to

$$S_{q_1 q_2}^{(3)}(\psi) = [TC_{q_1 q_2}^{(3)}(|\psi\rangle)]v[TC_{q_1 q_2}^{(3)}(|\psi\rangle)]^t. \quad (8)$$

Let $\lambda^2 = [(c_0 c_7 - c_1 c_6) - (c_2 c_5 - c_3 c_4)]^2 - 4(c_0 c_3 - c_1 c_2)(c_4 c_7 - c_5 c_6)$. Note that λ^2 is just the 3-tangle. The spectrum of $S_{1,2}^{(3)}(\psi)$ is $\pm\lambda, 0, 0$. We list the SJNFs of $S_{1,2}^{(3)}(\psi)$ and $S_{1,3}^{(3)}(\psi)$ in the Table IV. In light of Theorem 1, we can distinguish the six SLOCC classes of three qubits.

TABLE IV. SLOCC classification of three qubits

states	SJNF of $S_{1,2}^{(3)}(\psi)$	SJNF of $S_{1,3}^{(3)}(\psi)$
GHZ	$J_1(\pm\frac{1}{2})00$	$J_1(\pm\frac{1}{2})00$
W	$J_3(0)0$	$J_3(0)0$
A-BC	$J_2(0)J_2(0)$	$J_2(0)J_2(0)$
B-AC	$J_2(0)J_2(0)$	0000
C-AB	0000	$J_2(0)J_2(0)$
$ 000\rangle$	0000	0000

C. SLOCC classification of two qubits

For two qubits, Eq. (6) reduces to

$$S_{1,2}^{(2)}(\psi) = [TC_{1,2}^{(2)}(|\psi\rangle)][TC_{1,2}^{(2)}(|\psi\rangle)]^t. \quad (9)$$

The spectrum of $S_{1,2}^{(2)}(\psi)$ is 0, 0, 0, λ' , where $\lambda' = 2(a_0a_3 - a_1a_2)$. There are two cases for SJNFs. Case 1. For which $a_0a_3 = a_1a_2$ (it is a separate state), the SJNF of $S_{1,2}^{(2)}(\psi)$ is $J_2(0)00$. Case 2. For which $a_0a_3 \neq a_1a_2$ (it is an entangled state), the SJNF of $S_{1,2}^{(2)}(\psi)$ is $\lambda'000$. Thus, in light of Theorem 1, we can distinguish two-qubit states into two SLOCC classes.

IV. SLOCC CLASSIFICATION OF n QUBITS UNDER $\mathcal{A}_i \in SL(2, C)$

SLOCC classification under $A_i \in SL(2, C)$ or the classification under determinant one SLOCC operations was discussed in previous articles [3][16]. Note that under $\mathcal{A}_i \in SL(2, C)$, $G_1 \in SO(4, C)$ and Eq. (5) reduces to

$$S_{q_1q_2}^{(n)}(\psi') = G_1 S_{q_1q_2}^{(n)}(\psi) G_1^{-1}. \quad (10)$$

Thus, Eq. (10) leads to the following theorem.

Theorem 2. If the states $|\psi'\rangle$ and $|\psi\rangle$ of n qubits are SLOCC equivalent under $\mathcal{A}_i \in SL(2, C)$, then $S_{q_1q_2}^{(n)}(\psi')$ is orthogonally similar to $S_{q_1q_2}^{(n)}(\psi)$. The similarity implies that $S_{q_1q_2}^{(n)}(\psi')$ and $S_{q_1q_2}^{(n)}(\psi)$ have the same CP, spectrum, and SJNF up to the order of the standard Jordan blocks.

Example 4. $L_{ab3}^*(a=0)$ is SLOCC equivalent to $L_{ab3}(a=0)$ under $A_i \in SL(2, C)$ [25]. The SJNFs of $S_{1,2}^{(4)}$ are both $0b^2J_2(0)$.

Restated in the contrapositive the theorem reads: If two matrices $S_{q_1q_2}^{(n)}$ associated with two n -qubit pure states differ in their CPs, spectrums, or SJNFs,

TABLE V. Comparison between Theorems 1 and 2

	Theorem 1	Theorem 2
spect. ψ	$\lambda_1, \dots, \lambda_4$	$\lambda_1, \dots, \lambda_4$
spect. ψ'	$k\lambda_1, \dots, k\lambda_4$	$\lambda_1, \dots, \lambda_4$
SJNF ψ	$J_{\ell_1}(\lambda_1) \cdots J_{\ell_j}(\lambda_j)$	$J_{\ell_1}(\lambda_1) \cdots J_{\ell_j}(\lambda_j)$
SJNF ψ'	$J_{\ell_1}(k\lambda_1) \cdots J_{\ell_j}(k\lambda_j)$	$J_{\ell_1}(\lambda_1) \cdots J_{\ell_j}(\lambda_j)$

then the two states are SLOCC inequivalent under $\mathcal{A}_i \in SL(2, C)$. From Example 2, by Theorem 2 the two states ζ_4 and ζ_5 are SLOCC inequivalent under $A_i \in SL(2, C)$ because SJNFs of $S_{1,2}^{(4)}(\zeta_4)$ and $S_{1,2}^{(4)}(\zeta_5)$ are different.

Note that a SLOCC equivalent class may include infinite SLOCC equivalent classes under $\mathcal{A}_i \in SL(2, C)$.

V. CONCLUSION

In Theorem 1, we demonstrate that for two SLOCC equivalent states, the spectrums and SJNFs of the matrices $S_{q_1q_2}^{(n)}$ have proportional relationships. Invoking the proportional relationships, we partition pure states of n (≥ 4) qubits into 12 groups and 34 families under SLOCC, respectively.

In Theorem 2, we deduce that for two equivalent states under determinant one SLOCC operations, the spectrums, CPs, SJNFs of $S_{q_1q_2}^{(n)}$ are invariant. The invariance can be used for SLOCC classification of n qubits under determinant one SLOCC operations.

To make a comparison, we list the differences between Theorems 1 and 2 in Table V.

It is known that SJNF is used to solve a system of linear differential equations. The classification of SJNFs under SLOCC in this paper seems to be useful for classifying linear differential systems.

Acknowledgement—This work was supported by NSFC (Grant No. 10875061) and Tsinghua National Laboratory for Information Science and Technology.

VI. APPENDIX CORRESPONDING STATES OF FOUR QUBITS

Using G_{abcd} we obtain the following 8 states.

$G_1 = G_{abcd}(a = b = c = d \neq 0)$; (we will omit G_{abcd} next);

$G_2 : abcd \neq 0; b = c = d$ but $a \neq b$;

$G_3 : abcd \neq 0$, two of a, b, c , and d are equal while the other two are not equal;

G_4 : $abcd \neq 0$, a, b, c , and d consists of two pairs of equal numbers;

G_5 : $abcd \neq 0$, a, b, c , and d are distinct;

G_6 : only one of a, b, c , and d is zero and other three are equal;

G_7 : only one of a, b, c , and d is zero and only two of them are equal;

G_8 : only one of a, b, c , and d is zero and the other three are distinct.

Using L_{abc_2} we obtain the following 11 states.

$\zeta_1 = L_{abc_2}(a = b = c \neq 0)$;

$\zeta_2 = L_{abc_2}(abc \neq 0 \text{ and one of } a \text{ and } b \text{ equals } c)$;

$\zeta_3 = L_{abc_2}(abc \neq 0 \text{ and } a, b, c \text{ are distinct.})$;

$\zeta_4 = a(|0\rangle + |15\rangle) + b(|5\rangle + |10\rangle) + |6\rangle$, where $a \neq b$;

$\zeta_6 = L_{abc_2}$ (only one of a and b is zero while the other is equal to c);

$\zeta_7 = L_{abc_2}(c \neq 0 \text{ and only one of } a \text{ and } b \text{ is zero while the other is not equal to } c)$;

$\zeta_8 = L_{abc_2}(c = 0 \text{ and } a = b \neq 0)$; $\zeta_9 = L_{abc_2}(c \neq 0 \text{ and } a = b = 0)$; $\zeta_{10} = L_{abc_2}(c = 0 \text{ while } ab \neq 0 \text{ and } a \neq b)$; $\zeta_{11} = L_{abc_2}(c = 0 \text{ while only one of } a \text{ and } b \text{ is zero})$; $\zeta_{12} = L_{abc_2}(a = b = c = 0)$.

Using $L_{a_2b_2}$ we obtain the following two states.

$\eta_1 = L_{a_2b_2}(a = b \neq 0)$; $\eta_2 = L_{a_2b_2}(ab \neq 0 \text{ and}$

$a \neq b)$.

Let $L'_{ab_3} = b(|0\rangle + |15\rangle) + \frac{b+a}{2}(|5\rangle + |10\rangle) + \frac{b-a}{2}(|6\rangle + |9\rangle) + \frac{i}{\sqrt{2}}(|1\rangle + |2\rangle - |7\rangle - |11\rangle)$. Using L'_{ab_3} we obtain the following five states.

$\theta_1 = L'_{ab_3}(a = b \neq 0)$;

$\theta_2 = L'_{ab_3}(ab \neq 0 \text{ and } a \neq b)$;

$\theta_3 = a(|0\rangle + |15\rangle) + \frac{a}{2}(|5\rangle + |10\rangle + |6\rangle + |9\rangle) + \frac{i}{\sqrt{2}}(|1\rangle + |2\rangle - |7\rangle - |11\rangle)$ (obtained from $L'_{ab_3}(a = 0 \text{ but } b \neq 0)$);

$\theta_4 = L'_{ab_3}(b = 0 \text{ but } a \neq 0)$; $\theta_5 = L'_{ab_3}(a = b = 0)$.

Using L_{a_4} we obtain the following two states.

$\tau_1 = L_{a_4}(a \neq 0)$; $\tau_2 = L_{a_4}(a = 0)$.

Using $L_{a_20_3\oplus 1}$ we obtain the following one state.

$\kappa_1 = L_{a_20_3\oplus 1}(a \neq 0)$.

Let $L_{ab0_3\oplus 1} = \frac{a+b}{2}(|0\rangle + |15\rangle) + \frac{a-b}{2}(|3\rangle + |12\rangle) + |5\rangle + |6\rangle$. Using $L_{ab0_3\oplus 1}$ we obtain the following two states.

$\mu_1 = L_{ab0_3\oplus 1}(ab \neq 0 \text{ and } a \neq b)$; $\mu_2 = L_{ab0_3\oplus 1}(a = b \neq 0)$;

Let $\xi = \frac{a}{2}(|0\rangle + |3\rangle + |12\rangle + |15\rangle) + i|1\rangle - i|13\rangle + |10\rangle$. Using ξ we obtain the following two states.

$\xi_1 = \xi(a \neq 0)$; $\xi_2 = \xi(a = 0)$.

-
- [1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).
- [2] W. Dür, G. Vidal, and J.I. Cirac, Phys. Rev. A **62**, 062314 (2000).
- [3] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).
- [4] A. Miyake, Phys. Rev. A **67**, 012108 (2003).
- [5] Y. Cao and A. M. Wang, Eur. Phys. J. D **44**, 159 (2007).
- [6] D. Li, X. Li, H. Huang, and X. Li, Phys. Rev. A **76**, 052311 (2007).
- [7] O. Chterental and D.Z. Djoković, in Linear Algebra Research Advances, edited by G.D. Ling (Nova Science Publishers, Inc., Hauppauge, NY, 2007), Chap. 4, 133.
- [8] L. Lamata, J. León, D. Salgado, and E. Solano, Phys. Rev. A **75**, 022318 (2007).
- [9] D. Li, X. Li, H. Huang, and X. Li, Quantum Inf. Comput. **9**, 0778 (2009).
- [10] L. Borsten, D. Dahanayake, M. J. Duff, A. Mar-rani, and W. Rubens, Phys. Rev. Lett. **105**, 100507 (2010).
- [11] O. Viehmann, C. Eltschka, and J. Siewert, Phys. Rev. A **83**, 052330 (2011).
- [12] R.V. Buniy and T.W. Kephart, J. Phys. A: Math. Theor. **45**, 185304 (2012).
- [13] S.S. Sharma and N.K. Sharma, Phys. Rev. A **85**, 042315 (2012).
- [14] G. Gour and N.R. Wallach, Phys. Rev. Lett. **111**, 060502 (2013).
- [15] A. Wong and N. Christensen, Phys. Rev. A **63**, 044301, 2001.
- [16] J.-G. Luque and J.-Y. Thibon, Phys. Rev. A **67**, 042303 (2003).
- [17] M.S. Leifer, N. Linden and A. Winter, Phys. Rev. A **69**, 052304 (2004).
- [18] P. Levay, J. Phys. A: Math. Gen. **39**, 9533, (2006).
- [19] D. Z. Djoković and A. Osterloh, J. Math. Phys. **50**, 033509 (2009).
- [20] D. Li, X. Li, H. Huang, and X. Li, Phys. Rev. A **76**, 032304 (2007).
- [21] X. Li and D. Li, Phys. Rev. A **88**, 022306 (2013).
- [22] C. Eltschka, T. Bastin, A. Osterloh, and J. Siewert, Phys. Rev. A **85**, 022301 (2012).
- [23] V. Coffman, J. Kundu, and W.K. Wootters, Phys. Rev. A **61**, 052306 (2000).
- [24] X. Li and D. Li, Phys. Rev. Lett. **108**, 180502 (2012).
- [25] X. Li and D. Li, Phys. Rev. A **86**, 042332 (2012).
- [26] Hui Li, Shuhao Wang, Jianlian Cui, and Gui-Lu Long, Phys. Rev. A **87**, 042335 (2013).
- [27] Bo Li, Leong Chuan Kwek, and Heng Fan, J. Phys. A: Math. Theor. **45**, 505301 (2012).
- [28] X. Li and D. Li, Physical Review A **91**, 012302, (2015).
- [29] A. Osterloh and J. Siewert, Int. J. Quantum. Inform. **4**, 531 (2006).